Dollar Volume Discounts and Nonlinear Programming Quantity Discounts

by Rick Hesse, Feature Editor

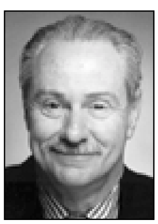
Those of us in production know the Wilson Lot-size formula (EOQ) and are familiar with using it for quantity discounts. Suppliers try to woo larger orders from customers by giving discount incentives to ease their own production scheduling. Consider two suppliers vying for your business that both sell three different products (1, 2, and 3) that have a minimum demand of 100, 152, and 200, respectively). Figure 1 shows the discount schedule and which supplier gives the best price.

Given this discount schedule, it makes sense to buy Product 1 and 3 from Beta Plugs, while Product 2 should be purchased from Delta Corporation, which yields a minimum cost of $68,672. Since the decision for each product is independent of the others, it is just a matter of choosing the minimum cost in E24:E25 for product 1, F24:F25 for product 2, and G24:G25 for product 3.

The critical part of the template in Figure 1 is the use of a vertical lookup table,

	A	B	C	D	E	F	G
1	VOLUME DISCOUNTS						
2	VOL	Product	Price	Quantity	Discount		
3				0	0.0%		
4		1	$105	50	2.0%		
5				100	3.0%		
6				0	0.0%		
7	Beta	2	$212	150	6.0%		
8	Plugs			300	8.0%		
9				0	0.0%		
10		3	$154	150	5.0%		
11				300	7.0%		
12				0	0.0%		
13		1	$110	60	5.0%		
14				150	7.0%		
15	Delta			0	0.0%		
16	Corp	2	$209	75	5.0%		
17				150	8.0%		
18				0	0.0%		
19		3	$160	100	5.0%		
20				200	8.0%		
21							
22		Quantity	Quantity	Quantity	Cost	Cost	Cost
23	Product	1	2	3	1	2	3
24	Beta	100	152	200	$10,185	$30,291	$29,260
25	Delta	100	152	200	$10,450	$29,227	$29,440
26	Orders	100	152	200	$68,672	Min Cost	

Figure 1: Quantity discount schedule.



Rick Hesse *is professor of quantitative methods at Pepperdine University in the Graziadio Graduate School of Business and Management. He received his BS, MS, and DSc at Washington University School of Engineering in applied math and computer science. Dr. Hesse is the author of* Managerial Spreadsheet Modeling & Analysis *and* Applied Management Science: A Quick & Dirty Approach *(with Gene Woolsey), articles in numerous journals, and software for personal computers. Rick was the first professor to be awarded the Outstanding Civilian Service Medal by the Department of the Army at West Point in 1982, and was the winner of the Decision Sciences Institute's Innovative Instructional Award in 1981.*

Dr. Rick Hesse
Graziadio Graduate School of Business and Management
Pepperdine University
Malibu, CA 90265
rickhesse@aol.com

	A	B	C	D	E	F	G	H	I	J
1	BUSINESS VOLUME DISCOUNTS									
2	$-Vol	Product	Price	$ Volume (000)	Discount					
3				$0	0.0%					
4	Beta	1	$105	$20,000	5.0%					
5	Plugs	2	$212	$35,000	6.5%					
6		3	$154	$75,000	8.0%					
7				$0	0.0%					
8	Delta	1	$110	$20,000	7.0%					
9	Corp	2	$209	$35,000	9.0%					
10		3	$160	$75,000	11.0%					
11										
12		Quantity	Quantity	Quantity	Cost	Cost	Cost		Discount	
13	Product	1	2	3	1	2	3	Totals	Price	w/Discount
14	Beta	100	0	62	$10,500	$0	$9,500	$20,000	95.0%	$19,000
15	Delta	0	152	138	$0	$31,768	$22,130	$53,898	91.0%	$49,047
16	Totals	100	152	200					Minimize =>	$68,047
17	Orders	100	152	200						

Figure 2: $-volume discount template.

illustrated by the formula in cell E24: =B24*(1-VLOOKUP(B24,D3:E5,2))*C4, which looks for the first value in the table D3:E5 greater than B24, then backs up a row and goes over to the second column for the discount amount and ends up with 100*(100%-2%)*$105 = $10,185. Lookup tables are one of the great advantages of spreadsheets, and this is a classic example of their usefulness (along with tax tables).

Dollar Volume Business Discounts

BellCore was an Edelman finalist in 1992 with their entry about business volume discounts (Sadrian, Hoadley, & Katz, 1993). They presented a problem similar to the one we have looked at, except that each supplier would give a discount based on the undiscounted dollar amount purchased for any of all three products. They developed a proprietary branch and bound program that found the minimum cost solution, and also allowed the user to add constraints such as number of necessary suppliers, and so forth. The $-volume discount schedule is shown in Figure 2, with all items ordered through Delta Corporation.

The VLOOKUP function is used to discount the total price (I14:I15) from each supplier to give the final amount owed each supplier in J14:J15, and thus the total in J16 becomes the objective function. This model is much more complicated than regular volume discounts because all the variables are now dependent and integer, the objective function is discontinuous, and the discounts are applied after each supplier total is determined.

Nonlinear Optimization Problem

The model in Figure 2 is a nonlinear optimization problem, with the objective function in J16 being minimized, and the integer variables are in B14:D15, adding up to the total order for each product (B16:D16), which must be ≥ to B17:D17. The Solver setup and options screen is shown in Figure 3 and the final results from running with all variables set to zero is what is shown in Figure 2. This result is achieved after running the Solver 3 times—twice with an error message: [* Filter does not support this file format | In-line.TIF *].

The problem with this model is the discontinuities caused by the price breaks. As the purchase amounts approach a break point, the objective function gets worse until it reaches the discount, and then drops dramatically. Figure 4 shows the total amounts for each supplier if all products were ordered from just one supplier. Choosing all products from Beta is more expensive than what the Solver found, but ordering all products from Delta is cheaper and was not found by the Solver.

However, nonlinear problems can be extremely sensitive to starting conditions. If we use the all Delta solution as our starting point and run the Solver, we get the same solution as shown in Figure 4 of $68,039. However, if we force the solution to spend at least $75,000 (add the constraint H15>=D10 in the Solver), then the global optimal solution is found, which is shown in Figure 5. This is $1,115 cheaper than ordering only the minimum required amount. By forcing the solution to another discount point, the Solver found the best combination of products to spend at least $75,000 and qualify for the higher discount.

However, we had to add a constraint to force the solution "up the hill" and over the "cliff." This is the problem with nonlinear functions that have many local optimal solutions. Imagine a hilly terrain shrouded in fog. The gradient search technique used by the Solver takes "baby" steps to determine the direction of steepest descent and then takes larger steps until it reaches the

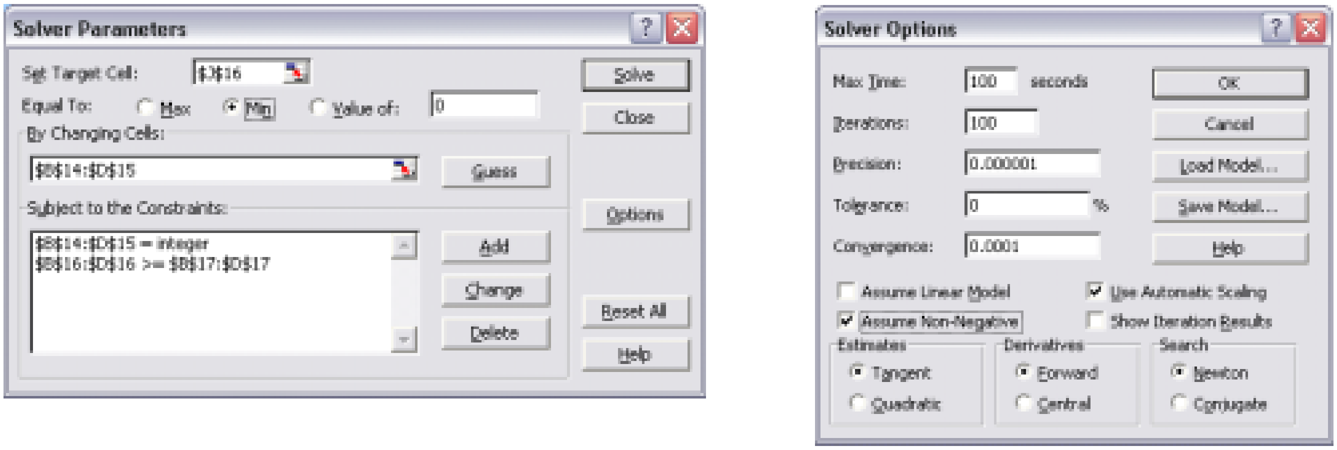


Figure 3: Solver setup for basic $-volume discount model.

Product	Quantity 1	Quantity 2	Quantity 3	Cost 1	Cost 2	Cost 3	Totals	Discount Price	w/Discount
Beta	100	152	200	$10,500	$32,224	$30,800	$73,524	93.5%	$68,745
Delta	100	152	200	$11,000	$31,768	$32,000	$74,768	91.0%	$68,039

Figure 4: Costs for single supplier.

	A	B	C	D	E	F	G	H	I	J
12		Quantity	Quantity	Quantity	Cost	Cost	Cost		Discount	
13	Product	1	2	3	1	2	3	Totals	Price	w/Discount
14	Beta	0	0	0	$0	$0	$0	$0	100.0%	$0
15	Delta	101	152	201	$11,110	$31,768	$32,160	$75,038	89.0%	$66,784
16	Totals	101	152	201					Minimize =>	$66,784
17	Orders	100	152	200						

Figure 5: Global optimal solution for $-volume discount.

bottom of a valley. Because of the fog, the Solver can’t see if there are any other valleys with a lower minimum. Therefore, the user must supply other starting points (and know something about the problem being solved) that might lead to a better solution.

Educational Premium Solver

For years, Frontline Systems, which provides the Solver Add-in for Excel, also provides an Educational Premium Solver with college textbooks that feature analysis using the Solver. This gives students and professors a “sneak peak” of a smaller version of the Premium Solver. One of the features is the Evolutionary Solver, which generates various starting values (each variable must have lower and upper bounds), and then “mutates” them in accordance with good or bad results. Although this Solver engine may take a lot longer to find solutions, it has a better chance of finding good ones or the optimal without a lot of manual assistance. The Solver setup is shown in Figure 6, with the addition of upper bounds on all the variables at 225 (lower bounds are non-negativity).

Figure 7 shows the solution when the starting point is all 0’s in a), which was the result of over 7,000 problems being run. When there is no further improvement, these starting values are used to run again. Because of the nature of the generation of

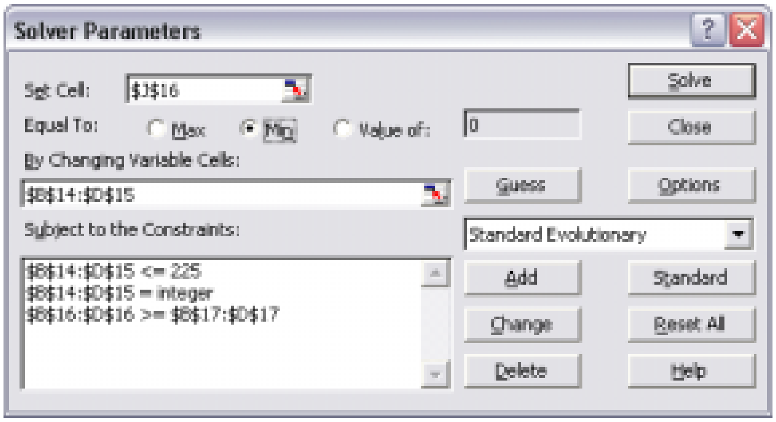


Figure 6: Educational Premium Solver.

a) Starting at 0's with EdPremSolver, over 7,000 trial solutions - not better than normal Solver

Product	Quantity 1	Quantity 2	Quantity 3	Cost 1	Cost 2	Cost 3	Totals	Discount Price	w/Discount
Beta	87	82	129	$9,135	$17,384	$19,866	$46,385	93.5%	$43,370
Delta	13	70	71	$1,430	$14,630	$11,360	$27,420	93.0%	$25,501
Totals	100	152	200					Minimize =>	$68,871
Orders	100	152	200						

b) Starting with answers in a), now there is real improvement

Product	Quantity 1	Quantity 2	Quantity 3	Cost 1	Cost 2	Cost 3	Totals	Discount Price	w/Discount
Beta	93	0	184	$9,765	$0	$28,336	$38,101	93.5%	$35,624
Delta	7	152	16	$770	$31,768	$2,560	$35,098	91.0%	$31,939
Totals	100	152	200					Minimize =>	$67,564
Orders	100	152	200						

Figure 7: Two runs with Evolutionary Solver.

starting points, this results in a very good solution, shown in b).

Finally, starting with each product being supplied by Delta at the minimum (100, 152, and 200), the Solver quickly finds the optimal solution by generating some solutions that spend at least $75,000 to get the highest discount rate from Delta, without having to add a constraint to force purchase of a least $75,000.

Conclusion

Dollar-volume discounts can save a lot of money for companies, but the mathematical optimization model is very nonlinear and discontinuous. Therefore it takes careful analysis to find the minimum cost, and using the Evolutionary Solver in the Premium Solver package can be extremely helpful. ■

Reference

Sadrian, A., Hoadley, B., & Katz, P. A. (1993). Improving the utility of the Bellcore Consortium. *Interfaces*, 23(1), Jan/Feb.

NOTE: *This article, including figures and Excel file, are available on the DSI website (see Decision Line) at www.decisionsciences.org.*