# A MODEL FOR THE SCHEDULING OF PROJECTS UNDER PENALTY AND REWARD ARRANGEMENTS

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## ABSTRACT

We develop a mixed integer linear programming model for scheduling projects under penalty and reward arrangements. Two variants of the model (model variants I and II) are also developed. Numerical examples given to compare the models show that the main model and its variants produce the same optimal values of project/activity durations, event times, bonuses/penalties, and total project costs. The model variants are much easier to solve than the main model. A remarkable finding is that the number of zero-one variables needed for the applications of model variants I and II are 3 and 4 respectively, irrespective of model size.

**Keywords:** zero-one linear programming, project scheduling, penalty and bonus clause.

## INTRODUCTION

The pioneering work of Kelly and Walker [11] has elicited a lot of research interests among many scholars and this has led to the developments and publications of different types of project scheduling models. These models are of different types and categories. Some of these include (1) client-contractor interaction problems (see Szmerekovskey [15], Ulusory et al. [17], and Westerney [22]), (2) Discount cash flows/maximization of net present value of projects (Etgar et al. [4], Ulusory et al. [17], and Vanhoucke et al. [21]), (3) resourced-constrained project-scheduling problems (see Mohring et al. [14], Tomos et al. [16], and Valls et al. [18] [19] ), (4) time-cost trade-off problems (Brucker [1], Grigoriev et al. [6], Jolayemi et al. [11], Kolish et al. [13], and Yang [23] ), and (5) time-dependent cash flows (Dayamand et al. [2], Herroelen et al. [7], and Vanhoucke et al. [20] [21], to mention a few).

Many important elements/sub-problems of the major problems listed above have also caught the attentions of scholars like Dayamand et al. [2], Etgar et al. [4], Gilbreath [5], Herroelen et al. [7], Jolayemi [8], Jolayemi et al. [9] [10] [11], Szmerekovskey [15], Vanhoucke et al. [21], and Westerney.

Penalty and reward arrangement is one of the most important sub-problem that can be used to achieve great success in the execution of a project. Penalty and reward clause is a provision in a contract that allows a contractor to be rewarded for completing a project before or on a stipulated due date and allows him to be penalized for failure to complete the project by such stipulated date. Besides Jolayemi [8] and Jolayemi et al. [9] [11], we have never come across any published research in which this clause is explicitly considered. Jolayemi et al. [11] developed three alternative models for scheduling projects under penalty and reward arrangements. From now henceforth, we will refer to the three alternative models by Jolayemi et al. [11] as J-P models.

The J-P models are developed for applications under three different special conditions and cases which are based on economic or profit considerations. Each model was developed for a single condition/case. This creates a great need for the developments of more comprehensive and efficient project scheduling models. That is, models that will not only be applicable under many conditions/cases but that will also be applicable for scheduling both profit and non-profit projects.

In this paper, we will develop a mixed integer programming (MILP) model for scheduling projects under penalty and reward arrangements. The development of the model will be based on the modification and extension of model II in Jolayemi et al. [11]. The new model will be applicable for scheduling for-profit and not-for-profit projects under more situations/cases than those in Jolayemi et al. (2007). Some variants of the model - variants that can be applied under some special conditions/cases that were not considered in Jolayemi et al. [11] – will be presented.

## THE GENERAL MILP MODEL AND ITS VARIANTS

### The General MILP Model

After some extensions and modifications of the J-P model (see Jolayemi and Pennington [11]), the general MILP model is developed and presented as follows.

$$\text{Minimize } z = \sum_{i,j\in G}(a_{ij} - h_{ij}x_{ij}) - \sum_{k=1}^{m} B_k y_k + \sum_{k=m}^{s} P_k w_k$$

Subject to:

$$T_i + x_{ij} - T_j \le 0 \quad \text{for all } (i,\ j) \in G \qquad \ldots\ldots\ldots\ldots\ldots\ldots(1).$$

$$x_{ij} \le L_{ij} \quad \text{for all } (i,\ j) \in G \qquad \ldots\ldots\ldots\ldots\ldots\ldots(2).$$

$$-x_{ij} \le -l_{ij} \quad \text{for all } (i,\ j) \in G \qquad \ldots\ldots\ldots\ldots\ldots\ldots(3).$$

$$T_n - T_1 \le \lambda \qquad \ldots\ldots\ldots\ldots\ldots\ldots(4).$$

$$\sum_{k=1}^{s-1} u_k = 1 \qquad \ldots\ldots\ldots\ldots\ldots\ldots(5).$$

$$T_n - \sum_{k=1}^{m} t_k y_k - \sum_{k=m}^{s} t_k w_k = 0 \qquad \ldots\ldots\ldots\ldots\ldots\ldots(6).$$

$$\sum_{k=1}^{m} y_k + \sum_{k=m}^{s} w_k = 1 \qquad \ldots\ldots\ldots\ldots\ldots\ldots(7).$$

$y_1 - u_1 \leq 0$ .......................(8).

$y_k - u_{k-1} - u_k \leq 0 \qquad k = 2, 3, ..........., m$ .........................(9).

$w_k - u_{k-1} - u_k \leq 0 \qquad k = m, m+1, .........., s-1$ .......................(10).

$w_k - u_{k-1} \leq 0 \qquad k = s$ ........................(11)..

$T_i, T_j, T_n, x_{ij}, \geq 0$ for all (i, j) $\in G$ and for i = 1, 2, …….., n:

$y_k \geq 0$ for $k = 1, 2, ........, m$; $w_k \geq 0$ for $k = m, m+1, ............., s$; and

$u_k$ = 0 or 1 for k = 1, 2, ……….., s-1;

where:

$a_{ij}$: the intercept made with the vertical axis by the linear approximation to the total cost curve. (See figures 3 and 4: the two figures are available on request).

$B_k$: the bonus or reward in dollars for completing the project in period $t_k$ (k = 1, 2, 3, ……, m and $t_m$ = d) before the expiration of the due date d. It is assumed that $B_k$ decreases as $t_k$ increases from the earliest project completion time E to the due date d.

d: the project due date.

$\Delta$: a very small fraction of time.

$\delta_{Bm}$: a very infinitesimal number substituted for the bonus $B_m$ ($B_m = 0$) at the time point $t_m$ (where $t_m$ is also the due date).

$\delta_{Pm}$: a very infinitesimal number substituted for the penalty $P_m$ ($P_m = 0$) at time $t_m$.

E: the earliest time that is technically possible for the completion of the project.

G: the set of all activities (i, j) of the project.

$h_{ij}$: the cost slope or gradient of the linear approximation to the total cost curve. (See figures 3 and 4).

(i, j): an activity that starts at node i and terminates at node j of the project network, i < j.

$L_{ij:}$ the normal time for activity (i, j).

$l_{ij}$: the "crash" time for activity (i, j).

$\lambda$: the constant constraint placed on the total project duration. It is the maximum allowable time for the completion of the project.

n: the end node of the project's network.

$P_{k:}$ the penalty in dollars for completing the project in period $t_k$ (k = m+1, m+2, ….., s and $t_s = \lambda$) after the expiration of the due date d. It is assumed that $P_k$ increases as $t_k$ increases from d to $\lambda$.

$P_{m+\Delta}$: the penalty in dollars for completing the project at the time point $t_m + \Delta$ after the due date d.

$T_i$: the start time for activity (i, j).

$T_j$: the completion time for activity (i, j).

$T_n$: the total project completion time (in weeks or months).

$T_l$: the earliest start time for the project.

$u_k$: a binary variable that is 1 if the project is completed within the time interval [$t_k$, $t_{k+1}$] and 0 otherwise.

$x_{ij}$: the duration (in weeks or months) for activity (i, j).

$y_k$; the variables that determine the value of the project completion time within an interval

($t_k$, $t_{k+1}$) and prorates bonus/penalty with respect to computed project completion time. $y_k$ (k = 1, 2, ……., m) take values in the closed interval [0, 1].

$w_k$: the variable that determines the value of the project completion time within an interval ($t_k$, $t_{k+1}$) and prorates the penalty with respect to the computed project completion time. $w_k$ (k= s, s+1, …….s) also takes value in the closed interval [0, 1].

The model is interpreted as follows.

The model's objective function minimizes the total cost of the project, including penalty and bonus costs.

Constraint (1) in the model ensures that the difference between the earliest event time $T_i$ and the latest event time $T_j$ must be at least as large as the activity duration $x_{ij}$.

The normal time $L_{ij}$ for the completion of an activity must be greater than or equal to the activity duration $x_{ij}$. Constraint (2) ensures this.

Constraint (3) expresses the fact that the duration of an activity must be greater than or equal to the "crash" time.

The time interval between the earliest time a project begins and the time it is completed must be less than or equal to a length of time $\lambda$. Constraint (4) ensures this.

Constraint (5) ensures that no two or more closed intervals are picked as the interval containing the optimal project completion time during any solution process.

Constraint (6) expresses $T_n$ as a linear combination of $t_k$'s (k = 1, 2, ………, s). The constraint makes it possible for the accurate value of project completion time to be determined within any closed interval [$t_k$, $t_{k+1}$] and for the right amount of bonus or penalty associated with the completion time to be automatically computed using the prorating variables $y_k$ (k = 1, 2, …….., m) and $w_k$ (k = m, m+1, ….. s).

Constraint (7) ensures that none of the variables $y_k$ (k = 1, 2, …….., m) and $w_k$ (k = m, m+1, ….. s) has a value greater than 1. It also ensures that the sum of any successive pair of $y_k$ or $w_k$ is not greater than 1.

Constraints (8), (9), (10), and (11) ensure that not more than two of the variables $y_k$ (k = 1, 2, …….., m) or $w_k$ (k = m, m+1, ….. s) are greater than zero. They also ensure that if two of the variables are greater than zero, then the two variables must be next to each other in the sequence $\{y_k\}_{k=1}^{m}$ or $\{w_k\}_{k=m}^{s}$. This makes it possible for $T_n$ to take any value in any closed interval [$t_k$, $t_{k+1}$] at optimality.

## VARIANTS OF THE MODEL

### Model Variant I

The assumptions underlining model variant I are that:

- Bonus decreases linearly within and across intervals from a maximum value at the time-point E to a minimum value of zero at the due date d.
- penalty increases linearly within and across intervals from a minimum value of zero at the due date d to a maximum value at a time point $\lambda$ after the due date.
- Bonus and penalty may decrease/increase at the same or different rates. In many cases, penalty will increase at a faster rate.

Based on the above assumptions and on the graphs in figures I and II (figures I and II are available on request) the model is developed and presented as follows.

$$\text{Minimize } w = \sum_{i,j\in G}(a_{ij} - h_{ij}x_{ij}) - B_1 y_1 - \delta_{Bm} y_m + \delta_{Pm} w_m + P_s w_s$$

Subject to:

$T_i + x_{ij} - T_j \le 0$ for all $(i, j) \in G$ ..........................................(1).

$x_{ij} \le L_{ij}$ for al $(i, j)$ ..........................................(2).

$-x_{ij} \le -l_{ij}$ for all $(i, j) \in G$ ..........................................(3).

$T_n - T_1 \le \lambda$ ..........................................(4).

$u_1 + u_m + u_{s-1} = 1$ ..........................................(5).

$y_1 + y_m + w_m + w_s = 1$ ..........................................(6).

$T_n - t_1 y_1 - t_m y_m - t_m w_m - t_s w_s = 0$ ..........................................(7).

$y_1 - u_1 \le 0$ ..........................................(8).

$y_m - u_1 - u_m \le 0$ ..........................................(9).

$w_m - u_m - u_{s-1} \le 0$ ..........................................(10).

$w_s - u_{s-1} \le 0$ ..........................................(11).

$T_1, T_j, T_n, x_{ij} \ge 0$ for all $(i, j) \in G$ and $i = 1, 2, \ldots, n$.

$y_1, y_m, w_m$ and $w_s \ge 0$; $u_1, u_m$, and $u_{s-1}$ are each 0 or 1.

**Model Variant II**

The following two assumptions and the assumptions stated earlier above apply here:

- a contractor is given some bonus for completing a project right on the due date, instead of no bonus.
- The contractor is awarded some penalty for failure to complete the project on the due date.

Based on these assumptions and on the graph in figure 2 (available on request), the model is developed and presented as:

$$\text{Minimize } z = \sum_{i,j\in G}(a_{ij} - h_{ij}x_{ij}) - B_1 y_1 - B_m y_m + \delta_{Pm} w_m + P_{m+\Delta} w_{m+\Delta} + P_s w_s$$

Subject to:

$T_i + x_{ij} - T_j \le 0$ for all $(i, j) \in G$ ..........................................(1).

$x_{ij} \le L_{ij}$ for al $(i, j)$ ..........................................(2).

$-x_{ij} \le -l_{ij}$ for all $(i, j) \in G$ ..........................................(3).

$T_n - T_1 \le \lambda$ ..........................................(4).

$y_1 + y_m + w_m + w_{m+\Delta} + w_s = 1$ ..........................................(5).

$u_1 + u_m + u_{m+\Delta} + u_{s-1} = 1$ ..........................................(6).

$T_n - t_1 y_1 - t_m y_m - t_{m+\Delta} w_{m+\Delta} - t_s w_s = 0$ ..............................................(7).

$y_1 - u_1 \leq 0$ ..............................................(8).

$y_m - u_1 - u_m \leq 0$ ..............................................(9).

$w_m - u_m - u_{m+\Delta} \leq 0$ ..............................................(10).

$w_{m+\Delta} - u_{m+\Delta} - u_{s-1}$ ..............................................(11).

$w_s - u_{s-1} \leq 0$ ..............................................(12).

$T_1, T_j, T_n, x_{ij} \geq 0 \text{ for all } (i, j) \in G \text{ and } i = 1, 2, \ldots\ldots\ldots, n$

$y_1, y_m, w_m + w_{m+\Delta}, w_{s-1} \geq 0;\ u_1, u_m, u_{m+\Delta}, \text{ and } u_{s-1} \text{ are each } 0 \text{ or } 1.$

## NUMERICA EXAMPLES AND COMPARISONS

We gave numerical examples to test and compare the main model and its variants. Optimal solution to each example was obtained using the LINDO solver. The examples produce good results. This shows that the three models work very well.

In each of the examples, the numbers of variables and constraints of model variants I and II respectively are much less than those of the main model. In each of the examples, the numbers of zero-one variables in model variants I and II are 3 and 4 respectively, irrespective of the model size, while the number of zero-one variables contained in the main model in the examples are more than three to five times as much.

The number of iterations before optimality is much larger for the main model than for each of its variants. Model variant I is smaller than model variant II in size and the number of iterations before optimality is always smaller for the former. The optimal solutions produced by the three models in each of the numerical examples are the same.

## CONCLUSIONS

The three models that are developed in this paper are useful and effective tools for planning for-profit, not-for-profit, and short- and long-term projects. Each of the models produces the same optimal values of project/activity duration and event time, bonus/penalty, and total project cost.

The conditions under which model variants I and II are applicable are more restrictive than those under which the main model is applicable. None-the-less, the two model variants have some remarkable advantages over the main model.

## REFERENCES

(Available on request).